Physics 137B (Professor Shapiro) Spring 2010

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Homework 8 Solutions

1. A photon of wavelength λ has energy $E = hc/\lambda$. So $N = \dfrac{1W}{(hc/\lambda)}$ photons are emitted per second from a 1W power source. The number of photons per unit area per unit time passing through a surface 10m away from the source and normal to the direction of propagation is then $n = N/(4\pi(10\text{m})^2)$.

(a) $\lambda = 10\text{m}$, $N = 5 \times 10^{25}\text{s}^{-1}$, $n = 4 \times 10^{22}\text{s}^{-1}\text{m}^{-2}$

(b) $\lambda = 0.1\text{m}$, $N = 5 \times 10^{23}\text{s}^{-1}$, $n = 4 \times 10^{20}\text{s}^{-1}\text{m}^{-2}$

(a) $\lambda = 5.89 \times 10^{-7}\text{m}$, $N = 2.97 \times 10^{18}\text{s}^{-1}$, $n = 2.36 \times 10^{15}\text{s}^{-1}\text{m}^{-2}$

(d) $\lambda = 10^{-10}\text{m}$, $N = 5 \times 10^{14}\text{s}^{-1}$, $n = 4 \times 10^{11}\text{s}^{-1}\text{m}^{-2}$

2. Averaging over all angular directions we have:

$$
\begin{aligned}
< \cos^2\theta > &= \frac{\int d\Omega \cos^2\theta}{\int d\Omega} \\
&= \frac{\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\cos^2\theta}{\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)} \\
&= \frac{2\pi \Big[\cos^3\theta/3\Big]_{\cos\theta=-1}^{\cos\theta=1}}{4\pi} \\
&= \frac{2\pi(2/3 - (-2/3))}{4\pi} \\
&= 1/3
\end{aligned}
$$

3. From section 11.3 of the text, the Einstein coefficients A and B are defined in relation to the rates of spontaneous emission ($W^{spont} = A$) and stimulated emission ($W^{stim} = B\rho$). So we have that:

$$\begin{aligned} \frac{W^{stim}}{W^{spont}} &= \frac{B\rho}{A} \\ &= \frac{1}{e^{\hbar\omega/kT} - 1} \end{aligned}$$

where equations 11.71 and 11.74a of the text have been used, and where ω is the frequency associated to the energy transition. In this case, $\hbar\omega = (-1/4 + 1)13.6\text{eV} = 10.2\text{eV}$ and $T = 2000K$, so that $kT = 0.1725\text{eV}$ and $\dfrac{1}{e^{\hbar\omega/kT} - 1} = 2.1 \times 10^{-26}$. Therefore:

$$\begin{aligned} W^{stim} &= \frac{1}{e^{\hbar\omega/kT} - 1} W^{spont} \\ &= (2.1 \times 10^{-26})(6.27 \times 10^{8}\text{s}^{-1}) \\ &= 1.3 \times 10^{-17}\text{s}^{-1} \end{aligned}$$

4. The half-life ($t_{1/2}$) is related to τ by:

$$\begin{aligned} 1/2 &= P(t_{1/2}) \\ &= \exp(-t_{1/2}/\tau) \\ \ln(1/2) &= -t_{1/2}/\tau \\ t_{1/2} &= \tau \ln(2) \end{aligned}$$

5. **(a)** The selection rules say that l must change by 1 and m must change by 0 or ± 1. So the following three decay routes are allowed by electric dipole transitions:

$$\begin{aligned} &|300> \rightarrow |2 \quad 1 \quad 1> \rightarrow |100> \\ &|300> \rightarrow |2 \quad 1 \quad 0> \rightarrow |100> \\ &|300> \rightarrow |2 \quad 1 \quad -1> \rightarrow |100> \end{aligned}$$

(b) From equation 11.80 of the text, the spontaneous emission rate is $W^s_{ab} = \frac{4}{3}\frac{\alpha}{e^2c^2}\omega^3_{ba}|\mathbf{D}_{ab}|^2$ where:

$$\begin{aligned}
\mathbf{D}_{ab} &= -e\int_0^\infty drR_{21}(r)R_{30}(r)r^3\int d\Omega Y^*_{1m}(\theta,\phi)\hat{\mathbf{r}}Y_{00}(\theta,\phi) \\
&= -e\int_0^\infty drR_{21}(r)R_{30}(r)r^3(\frac{1}{\sqrt{6}}(-\delta_{m,1}+\delta_{m,-1})\hat{\mathbf{x}} + \frac{1}{\sqrt{6}}i(\delta_{m,1}+\delta_{m,-1})\hat{\mathbf{y}} \\
&\qquad + \frac{1}{\sqrt{3}}(\delta_{m,0})\hat{\mathbf{z}}
\end{aligned}$$

(using equation 11.85 of the text)

So the transition rate is:

$$\begin{aligned}
W^s_{ab} &= \frac{4}{3}\frac{\alpha}{c^2}\omega^3_{ba}\Big|\int_0^\infty drR_{21}(r)R_{30}(r)r^3\Big|^2(\frac{1}{6}(\delta_{m,1}+\delta_{m,-1}) + \frac{1}{6}(\delta_{m,1}+\delta_{m,-1}) \\
&\qquad + \frac{1}{3}(\delta_{m,0})) \\
&= \frac{4}{9}\frac{\alpha}{c^2}\omega^3_{ba}\Big|\int_0^\infty drR_{21}(r)R_{30}(r)r^3\Big|^2(\delta_{m,1}+\delta_{m,-1}+\delta_{m,0})
\end{aligned}$$

The above formula shows that the transition rate is equal to each value of m. Thus there is a 1/3 probability of decaying through any of the three channels, and 1/3 of the atoms would decay via each of the routes in part (a).

(c) We have that

$$\begin{aligned}
&\int_0^\infty drR_{21}(r)R_{30}(r)r^3 \\
&= \int_0^\infty dr\frac{2}{\sqrt{3}}(1/6a_\mu^2)^{3/2}(r/a_\mu)(1-2r/3a_\mu+2r^2/27a_\mu^2)\exp(-r/2a_\mu-2/3a_\mu)r^3 \\
&= \frac{2^7 3^4}{5^6}\sqrt{2}a_\mu
\end{aligned}$$

and also that

$$\begin{aligned}
\omega_{ba} &= (E_3-E_2)/\hbar \\
&= \frac{5}{36}E_1/\hbar \\
&= \frac{5}{72}\frac{\mu c^2}{\hbar}\alpha^2.
\end{aligned}$$

So the transition rate from the $|300>$ state to the $|21m>$ state is (using the expression in part (b)):

$$\begin{aligned}
W_{ab}^{s} &= \frac{4}{9}\frac{\alpha}{c^2}\left(\frac{5}{72}\frac{\mu c^2}{\hbar}(\alpha)^2\right)^3\left|\frac{2^7 3^4}{5^6}\sqrt{2}a_\mu\right|^2(\delta_{m,1}+\delta_{m,-1}+\delta_{m,0}) \\
&= \frac{4}{9}\frac{\alpha}{c^2}\left(\frac{5}{72}\frac{\mu c^2}{\hbar}(\alpha)^2\right)^3\left|\frac{2^7 3^4}{5^6}\sqrt{2}(\hbar/\mu\alpha c)\right|^2(\delta_{m,1}+\delta_{m,-1}+\delta_{m,0}) \\
&= \frac{2^8}{5^9}\frac{\alpha^5 c^2\mu}{\hbar}(\delta_{m,1}+\delta_{m,-1}+\delta_{m,0}) \\
&\approx 2.1\times 10^6\mathrm{s}^{-1}(\delta_{m,1}+\delta_{m,-1}+\delta_{m,0})
\end{aligned}$$

So the transition rate to each of the three final states ($m=-1,0,1$) is $2.1\times 10^6\mathrm{s}^{-1}$. So the total decay rate of the $|300>$ state is $W = 3\times(2.1\times 10^6\mathrm{s}^{-1}) = 6.3\times 10^6\mathrm{s}^{-1}$, leading to a lifetime of $\tau = 1/W = 1.6\times 10^{-7}\mathrm{s}$.